Intermittency and anomalous scaling of passive scalars in any space dimension

Gregory L. Eyink
Department of Mathematics, University of Arizona, Tucson, Arizona 85721

(Received 18 March 1996)

We establish exact inequalities for the structure-function scaling exponents of a passively advected scalar in both the inertial-convective and viscous-convective ranges. These inequalities involve the scaling exponents of the velocity structure functions and, in a refined form, an intermittency exponent of the convective-range scalar flux. They are valid for three-dimensional Navier-Stokes turbulence and satisfied within errors by present experimental data. The inequalities also hold for any "synthetic" turbulent velocity statistics with a finite correlation in time. We show that for time-correlation exponents of the velocity smaller than the "local turnover" exponent, the scalar spectral exponent is strictly less than that in Kraichnan's [Phys. Rev. Lett. **72**, 1016 (1994); Phys. Fluids **11**, 945 (1968); J. Fluid Mech. **64**, 737 (1974); **62**, 305 (1974)] soluble "rapid-change" model with velocity δ correlated in time. Our results include as a special case an exponent inequality derived previously by Constantin and Procaccia [Nonlinearity **7**, 1045 (1994)], but with a more direct proof. The inequalities in their simplest form follow from a Kolmogorov-type relation for the turbulent passive scalar valid in each space dimension d. Our improved inequalities are based upon a rigorous version of the refined similarity hypothesis for passive scalars. These are compared with the relations implied by "fusion rules" hypothesized for scalar gradients. [S1063-651X(96)04308-5]

PACS number(s): 47.27.−i

I. INTRODUCTION

Much progress has been made recently in the understanding of anomalous scaling for the problem of randomly advected scalars [1–10]. The dynamical equation of the model is

$$[\partial_t+\mathbf{v}(\mathbf{r},t)\cdot\nabla_{\mathbf{r}}]\theta(\mathbf{r},t)=\kappa\triangle_{\mathbf{r}}\theta(\mathbf{r},t)+f(\mathbf{r},t), \tag{1}$$

with $\theta(\mathbf{r},t)$ the scalar field, $f(\mathbf{r},t)$ a (stochastic or deterministic) source, and $\mathbf{v}(\mathbf{r},t)$ a random incompressible velocity field. The issue of interest is the presumed scaling law

$$S_p(l)\sim l^{\zeta_p(\theta)} \tag{2}$$

as $l\to 0$ for the scalar structure functions $S_p(l;\theta)\equiv\langle|\Delta_l\theta|^p\rangle$. For a high Reynolds number turbulence and for molecular diffusivity of the order of magnitude of the molecular viscosity, or greater, $\kappa\geqslant\nu$, the dimensional theory of Obukhov [11] and Corrsin [12] implies that,

$$S_p(l)\sim(\chi/\varepsilon^{1/3})^{p/2}l^{p/3}, \tag{3}$$

in which $\chi=\kappa\langle[\nabla\theta]^2\rangle$ is the (mean) scalar dissipation and ε is the dissipation of kinetic energy. The scaling law Eq. (3) is supposed to hold for $L\gg l\gg(\kappa^3/\varepsilon)^{1/4}$, where L is the length scale of the scalar source, assumed less than the integral scale of velocity. This specifies the *inertial-convective range*. Thus $\zeta_p(\theta)=p/3$ in the classical theory of this range. On the other hand, for $\kappa\ll\nu$ there is another range, $L\gg l\gg(\kappa^3/\varepsilon)^{1/4}$, in which it is now assumed that L is at, or smaller than, the Kolmogorov dissipation scale $(\nu^3/\varepsilon)^{1/4}$. Over this range, the so-called *viscous-convective range*, the theory of Batchelor [13] proposes that

$$S_p(l)\sim(\chi/\sigma)^{p/2}\log^{x_p}(l/L), \tag{4}$$

with σ a mean shear strength. Here $\zeta_p(\theta)=0$ formally for all p. Whereas [11,12] predicted $\zeta_p(\theta)$ to be a linear function of index p, it is now generally expected that these exponents are some nontrivial concave functions of p, i.e., that there is anomalous scaling [14].

The recent work on this problem cited above mostly deals with a special model in which the Eulerian velocity field is zero-mean Gaussian, δ correlated in time

$$\langle v_i(\mathbf{r},t)v_j(\mathbf{r}',t')\rangle=V_{ij}(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{5}$$

This so-called "rapid-change model" was investigated by Kraichnan [15], who observed that for this case the infamous closure problem is absent: the Nth-order correlator of θ obeys equations depending only upon itself and *lower-order* correlators. The recent analytical investigations explore particular limiting regimes: the case with space dimension $d\gg 1$ in [6,7] and the case with eddy-diffusivity exponent $0<\zeta\ll 1$ in [4,5]. The latter exponent is defined by the assumed scaling relation for the (Richardson) eddy-diffusivity tensor

$$K_{ij}(\mathbf{r})\equiv\frac{1}{2}\int_{-\infty}^{t}ds\langle[v_i(\mathbf{r},t)-v_i(\mathbf{0},t)][v_j(\mathbf{r},s)-v_j(\mathbf{0},s)]\rangle$$
$$=V_{ij}(\mathbf{0})-V_{ij}(\mathbf{r}), \tag{6}$$

that

$$K_{ij}(\mathbf{r})\sim D(d-1)\delta_{ij}r^{\zeta}+D\zeta r^{\zeta}\left(\delta_{ij}-\frac{r_ir_j}{r^2}\right) \tag{7}$$

for small r. Note that Eq. (6) is an analog for this model of Taylor's 1921 exact formula for the eddy-diffusivity [16] (which involves instead Lagrangian velocities).

It is our purpose here to consider the problem with a finite time correlation of the convecting velocity field. This includes the realistic case where the velocities are turbulent solutions of Navier-Stokes dynamics. In addition, our results apply to a model recently considered [8] with the Eulerian velocity field taken as a Gaussian with covariance obeying dynamical scaling

$$V_{ij}(\mathbf{r},t)\equiv\frac{1}{2}\langle[v_i(\mathbf{r},t)-v_i(\mathbf{0},t)][v_j(\mathbf{r},0)-v_j(\mathbf{0},0)]\rangle$$
$$=\frac{Dr^{\zeta}}{\tau_r}\left[\delta_{ij}g_{\parallel}\left(\frac{t}{\tau_r}\right)+\left(\delta_{ij}-\frac{r_ir_j}{r^2}\right)g_{\perp}\left(\frac{t}{\tau_r}\right)\right], \quad (8)$$

with $\tau_r=\tau_L(r/L)^z$. Our results in the finite correlation-time models shall be applicable to both the limiting regimes studied for the rapid-change model: $d\gg 1$ and $0<\zeta\ll 1$. However, rather than asymptotic formulas for the scaling exponents, we shall derive exact inequalities. One interest of our results is that they point up some significant differences between the zero and finite correlation-time problems.

Our simplest set of inequalities are based upon the relation

$$\langle\Delta_l v_{\parallel}[\Delta_l\theta]^2\rangle=-\frac{4}{d}\chi l. \quad (9)$$

This equation is valid for $L\gg l\gg\eta_D$, where L is the length scale of the scalar source and η_D is a dissipation length, given by the Obukhov-Corrsin length $(\kappa^3/\varepsilon)^{1/4}$ [11,12] for high Reynolds number Navier-Stokes turbulence and by $(\kappa/D)^{1/\zeta}$ for the model of [8] [Eq. (8)]. Equation (9) is a relation analogous to that of Kolmogorov for the third-order velocity structure function [17] and it was proved for $d=3$ by Yaglom in 1949 [18]. It simply expresses the constancy of scalar flux over the convective range of scales. By direct application of the Hölder inequality, we will derive from this relation a basic set of inequalities relating the scaling exponents of pth structure functions of the scalar $\zeta_p(\theta)$ with those of the velocity $\zeta_p(v)$. The implications of these results will be discussed in Sec. II.

Furthermore, we shall derive an improved set of inequalities by means of a *refined similarity relation* (RSR) for the passive scalar. In a precise version stated below, the RSR we prove is

$$X_l(\mathbf{r})\sim\frac{\Delta_l v(\mathbf{r})[\Delta_l\theta(\mathbf{r})]^2}{l}, \quad (10)$$

in which $X_l(\mathbf{r})$ is a local scalar flux to scales less than l at space point $\mathbf{r}$. (cf [14].) The corresponding inequalities will involve the *intermittency exponent* $\tau_p(X)$ of the scalar flux

$$\langle|X_l|^p\rangle\sim l^{\tau_p(X)}, \quad (11)$$

which measures the increasing spatial spottiness of the flux as $l\to 0$. These inequalities rigorously establish an intuitive fact: that convective-range intermittency of the scalar flux implies anomalous scaling of the scalar structure functions. These results are given in Sec. III, along with some general discussion of refined similiarity hypotheses for passive scalars, including the relation to ''fusion rules'' proposed for scalar gradients.

II. YAGLOM RELATION INEQUALITIES

We shall sketch here very concisely the proof of the Yaglom relation for any space dimension d. See also [19] and, for more details [20], and Appendix II of [21]. The first step is to define a (mean) ''physical space scalar flux'' via

$$X(\mathbf{l})\equiv-\frac{1}{2}\frac{d}{dt}\langle\theta(\mathbf{r},t)\theta(\mathbf{r}+\mathbf{l},t)\rangle|_{\mathrm{conv},t=0}. \quad (12)$$

The subscript ''conv'' indicates that only the convective terms in Eq. (1) are used. A simple calculation using incompressibility and spatial homogeneity gives $X(\mathbf{l})=-1/4\nabla_{\mathbf{l}}\cdot\langle\Delta_{\mathbf{l}}\mathbf{v}[\Delta_{\mathbf{l}}\theta]^2\rangle$. Assuming also spatial isotropy, the vector $\mathbf{A}(\mathbf{l})=\langle\Delta_{\mathbf{l}}\mathbf{v}[\Delta_{\mathbf{l}}\theta]^2\rangle$ is (for $d>2$) of the form $\mathbf{A}(\mathbf{l})=A_{\parallel}(l)\hat{\mathbf{l}}$, where $\hat{\mathbf{l}}$ is the unit vector in the direction of $\mathbf{l}$. In the convective range of length scales l with constant mean scalar flux, Eq. (12) becomes $-4\chi=\nabla_{\mathbf{l}}\cdot\mathbf{A}(\mathbf{l})$ or

$$-4\chi=\frac{d-1}{l}A_{\parallel}(l)+\frac{dA_{\parallel}}{dl}(l). \quad (13)$$

The only solution of this equation regular for $l\to 0$ is

$$A_{\parallel}(l)=-\frac{4\chi}{d}l. \quad (14)$$

This completes the derivation of Eq. (9). It is useful to remark here that the Yaglom relation does *not* hold in the Kraichnan rapid-change model. In fact, the left-hand side of Eq. (9) is not even well defined in Kraichnan's model, since it is expressed by a product of $\Delta_l v_{\parallel}$ and $[\Delta_l\theta]^2$ *at a single instant*. However, with the δ correlation in time, the velocity is a distribution-valued (generalized) process and the single-time values are not defined. The analogous result for Kraichnan's model is [15]

$$S_2(l)\sim\frac{\chi}{Dd^2}l^{2-\zeta}, \quad (15)$$

which, like the Yaglom relation for the finite correlation-time velocity statistics, is an exact result in the δ correlated model.

A simple set of exponent inequalities follows from the Yaglom relation as a straightforward application of the Hölder inequality. The inequalities for $\zeta_p(\theta)$ involve as well the exponents $\zeta_p(v)$ of the (absolute) structure functions of velocity $S_p(l;v)\equiv\langle|\Delta_l v|^p\rangle\sim l^{\zeta_p(v)}$. They are simplest to state in terms of the exponents $\sigma_p(\theta)=\zeta_p(\theta)/p$ and $\sigma_p(v)=\zeta_p(v)/p$. By the Yaglom relation and the Hölder inequality,

$$4\chi l=|\langle\Delta_l v_{\parallel}[\Delta_l\theta]^2\rangle|$$
$$\leqslant\langle|\Delta_l\mathbf{v}|^q\rangle^{1/q}\langle|\Delta_l\theta|^p\rangle^{2/p}\sim l^{2\sigma_p(\theta)+\sigma_q(v)} \quad (16)$$

for

$$p\geqslant 2,\quad \frac{2}{p}+\frac{1}{q}=1. \quad (17)$$

As this relation must hold in the infinitely long convective range as $l\to 0$, it follows that

$$2\sigma_p(\theta)+\sigma_q(v)\leq 1. \tag{18}$$

We have used the isotropic form of the Yaglom relation, but this is inessential (see [21]). The special case of Eq. (18) for $p=\infty, q=1$ was previously derived by Constantin and Procaccia [22] on the basis of estimates for Hausdorff dimensions of scalar level sets; see Eq. (4.2) there. The authors of [22] were not specific about which order p,q of exponents were involved, but inspection of the proof shows that $p=\infty$ and $q=1$, in our notation, was used. Actually, most of their proof generalizes to general p,q satisfying Eq. (17), except for the result on dimensions of level sets E_θ, $D_H(E_\theta)\leq d-\sigma_\infty(\theta)$, their Eq. (1.8), which used $p=\infty$. Incidentally, we disagree with the conclusion of [22] that the estimate is sharp, i.e., an equality. This claim in Sec. IV is based on an opposite inequality $2\sigma_\infty(\theta)+\sigma_1(v)\geq 1$, supposed to be derived in [23], which we dispute. In fact, examination of the argument of [23] shows that an implicit assumption was made that a *spatially uniform* diffusive cutoff exists, below which length scale the scalar graph is smooth. This assumption was used in the selection of r_0 in their inequality Eq. (4.23). To bound $\mathrm{vol}_d(G(B))$ from *below* requires that $G(B)$ is smooth on scales less than or equal to r_0 so that a uniform choice of r_0 may be made. It is possible, however, that there is a ''local fluctuating cutoff,'' as postulated in some multifractal pictures; see [24]. In that case, the uniform choice of r_0 would fail and the result, as well as its proof, might break down. Our very simple derivation here shows that Eq. (18) belongs to a family of inequalities that are a consequence just of the constancy of scalar flux. These inequalities express a complementarity between the regularity of the velocity and scalar fields: if the (Besov) regularity exponent of velocity $\sigma_q(v)$ is ''big'' then the corresponding scalar exponent $\sigma_p(\theta)$ must be ''small.''

The basic inequality Eq. (18) may also be written as

$$\zeta_p(\theta)\leq \frac{p}{2}[1-\sigma_q(v)] \tag{19}$$

for $p\geq 2, 2/p+1/q=1$. This relation may be considered under various special assumptions. If the scaling exponents of velocity v are taken to be K41, i.e. $\sigma_q(v)=1/3$ for all $q\geq 1$; then it follows that

$$\zeta_p(\theta)\leq \frac{p}{3} \tag{20}$$

for $p\geq 2$. Thus the Obukhov-Corrsin predictions for the inertial-convective range appear as upper bounds. Likewise, if the velocity field is assumed smooth, or $\sigma_q(v)=1$ for all $q\geq 1$, then

$$\sigma_p(\theta)\leq 0. \tag{21}$$

The smoothness assumption would hold, for example, for the velocity field in the viscous dissipation range and then the Batchelor exponents for the viscous-convection range appear as upper bounds. More generally, if it is assumed that $\sigma_q(v)=h$ for all $q\geq 2$, then

$$\zeta_p(\theta)\leq \frac{p}{2}(1-h). \tag{22}$$

This assumption on the velocity scaling exponents corresponds to a ''monofractal'' field and would be true for convection by a Gaussian random velocity field.

The last inequality has some interesting consequences for the model studied in [8]. In that model the velocity field is space-time Gaussian with covariance satisfying dynamical scaling Eq. (8). Setting $t=0$ in that equation, it is easy to see that $2h=\zeta-z$ or

$$\zeta=2h+z. \tag{23}$$

As pointed out above, ζ has roughly the interpretation of an eddy-diffusivity exponent analogous to the Richardson exponent $\zeta_R=4/3$ [25]. This is not entirely accurate since the exponent appears in the scaling law Eq. (8) hypothesized for *Eulerian velocities* in the model of [8]. For $z<1$ Eq. (8) is not an accurate representation of Eulerian time correlations, which will then be dominated by convective sweeping. Nevertheless, keeping to this terminology, there is also for fixed ''eddy diffusivity exponent'' ζ a complementarity between the magnitudes of velocity regularity exponent h and dynamical scaling exponent z: if one is big, the other is small. Of course, this is just due to the simple heuristic that eddy-diffusivity $K_l\sim v_l^2\tau_l$. A similar relation should also hold for Navier-Stokes turbulence, i.e. $\zeta=\zeta_2(v)+z$, except that there the scaling exponent z will correspond to *Lagrangian* time correlations. It is only for Lagrangian time functions that the dynamical scaling can hold and, furthermore, the exact Taylor formula Eq. (7) involves such correlations. By means of Eq. (23), the main inequality Eq. (19) may be reexpressed as

$$\zeta_p(\theta)\leq \frac{p}{4}(z+\gamma), \tag{24}$$

where $\gamma\equiv 2-\zeta$. In fact, Eq. (23) states that

$$1-h=\frac{z+\gamma}{2}, \tag{25}$$

so that it is a direct consequence. Equation (24) holds for all $p\geq 2$ in the model of [8]; furthermore, it will hold also for $p=2$ in Navier-Stokes turbulence, if $\zeta=\zeta_2(v)+z$ as expected.

In [8] an expansion of the rapid-change model was developed in the magnitude ϵ of the correlation time. They employed a particular choice of dynamical exponent $z=\gamma$. Their plausible physical argument for this choice ran as follows: the ''local turnover time'' of scalar eddies at scale l is $t_l\sim l/v_l\sim l^{1-h}$ and this defines a ''local turnover value'' of the dynamical exponent $z=1-h$. Note that by using Eq. (23) this value is achieved precisely when $\zeta=1+h$ or $\gamma=1-h$. In other words, the local turnover exponent is obtained when $z=\gamma$. For $z>\gamma$ the velocity at vanishingly small scales changes randomly at a faster and faster rate relative to the evolution time of the scalar eddies. Hence it is plausible that the predictions of the rapid-change model will hold in that case. On the other hand, for $z<\gamma$, the realizations of the velocity field are selected randomly at a rate that goes to zero compared to the scalar cascade rate, i.e., the velocity field randomness is ''frozen in.'' We shall now observe that in the latter case of quenched randomness, or $z<\gamma$, the scalar spectral exponent

$$\zeta_2(\theta)<\gamma, \tag{26}$$

with strict inequality. This is direct from Eq. (24). More generally, for $p\geqslant 2$

$$\zeta_p(\theta)<\frac{p}{2}\gamma. \tag{27}$$

Hence all of the ζ_p's for $p\geqslant 2$ are *strictly* smaller than the ''classical'' values. This is not so surprising for the higher-p values $p>2$, since this is a familiar situation usually associated to ''intermittency.'' The present strict inequalities, including the unusual case $p=2$, have a different origin. They arise just from the condition of constant mean flux, which requires smaller $\zeta_p(\theta)$'s when h is bigger. However, z smaller than the local turnover exponent at fixed value of eddy-diffusivity exponent ζ requires h bigger than its classical value $1-\gamma$. Indeed, $1-\gamma<h<1-z$ from Eq. (25) if $z<\gamma$.

Another important comparison between the rapid-change model and the finite time-correlation cases arises from the dimension dependence of the Yaglom relation. It is clear from Eq. (15) for $S_2(l)$ in the rapid-change model that it goes to a finite limit as $d\to\infty$ if and only if $D\propto D_0/d^2$, with D_0 fixed. In fact, it follows from the work of Chertkov *et al.* in [6] that with that choice of d dependence of D all scalar correlations have a nontrivial limit as $d\to\infty$ and, in fact, correspond to the correlations of a Gaussian field. Note that this dependence of D implies that each of the $i=1,\dots,d$ components of the velocity field $v_i(l)$ at scale l has typical magnitude $l^{1-\gamma}/\sqrt{d}$ and, likewise, each of the d^2 components of the strain tensor $\boldsymbol{\sigma}_l=1/2\,[(\nabla\mathbf{v}_l)+(\nabla\mathbf{v}_l)^\top]$ are of order $l^{-\gamma}/\sqrt{d}$. It may be argued on the basis of theory of random matrices that the typical strains along principal axes, the eigenvalues of $\boldsymbol{\sigma}_l$, are order $l^{-\gamma}$ as $d\to\infty$; see [26,27]. This seems to be the correct scaling for a nontrivial limit, since the strain magnitudes gives the rate $\sim 1/t_l$ of scalar cascade. [Actually, the *principal eigenvalue* of σ_l may be expected to determine the rate of scalar cascade; see [26]. However, for random Wigner matrices with asymptotic semicircle distribution of eigenvalues, it is known also that the leading eigenvalue is within $O(1/d^{2/3})$ of the right edge of the spectrum.] The Yaglom relation shows that the matter is not so simple for the finite time-correlation situation. For that relation a scaling $v\sim 1/d$ is required to obtain a finite limit as $d\to\infty$. This does not contradict the results of [8], since they take $\epsilon=\tau_l/t_l$ as the small parameter of their expansion. Working this through, one finds that this amounts to taking $v_l\sim(D_0/\sqrt{\epsilon d})l^{1-\gamma}$ and $\tau_l\sim\epsilon(l^\gamma/D_0)$. For any d, this correctly recovers the δ correlated model in the limit $\epsilon\to 0$. In fact, ϵ is just the quantity denoted τ_* in [15]. For the validity of the Yaglom relation it is therefore required that $\epsilon\sim d$, which is clearly incompatible with the condition of [8] that $\epsilon\ll 1$ at large d. (The proportionality of single-time velocity realizations to $\epsilon^{-1/2}$ here, as well as in Kraichnan's original 1968 derivation, makes clear that such single-time values do not exist in the δ correlated model obtained by the limit $\epsilon\to 0$. Only integrals over some finite time interval are well defined and the left-hand side of the Yaglom relation, for the idealized white-noise limit, is a meaningless expression.)

III. REFINED SIMILARITY INEQUALITIES

We shall now derive inequalities that improve those from the Yaglom relation. The basic idea of the proof is a scaling relation between the *local scalar flux* variable $X_l(\mathbf{r})$ and the difference variables of velocity and scalar at the same point $\mathbf{r}$ [Eq. (10)]. This is an analog of the *refined similarity hypothesis* (RSH) in three dimensions, which, in the version of Kraichnan [29], states that local energy flux scales as $\Pi_l(\mathbf{r})\sim[\Delta_l v(\mathbf{r})]^3/l$ in terms of the velocity difference at the same point. The proofs given below follow closely methods used in our discussion of the three-dimensional (3D) RSH in [28] and the 2D RSH for vorticity scaling exponents in [21]. If we assume

$$\langle|X_l|^p\rangle\sim l^{\tau_p(X)}, \tag{28}$$

then there follow heuristically from Eq. (10) relations between the exponents $\zeta_p(\theta)$, $\zeta_q(v)$, and $\tau_r(X)$. The exponents $\tau_p(X)$ measure the increasing spatial intermittency or ''spottiness'' of the scalar flux at decreasing length scales. In fact, since $\langle X_l\rangle=\chi$ over the long convective interval of l, it may be expected that the (concave in p) exponent $\tau_p(X)$ is *negative* for monents $p>1$. The corresponding growth in moments of X_l as $l\to 0$ reflects the increase in its fluctuations. As we shall establish below, the intermittency of the convective range scalar flux implies anomalous scaling of the scalar structure functions over that same interval. It is this connection between intermittency and anomalous scaling that is the essence of Kraichnan's RSH [29]. After these results are derived as theorems below, we shall comment on the relationship with other refined similarity hypotheses for passive scalars recently proposed [30,31,19]. These latter hypotheses are motivated by the original Kolmogorov RSH [32], which involves space-averaged dissipation rather than local flux.

We first must introduce an appropriate definition of the local scalar flux. It is most easily done using a smooth *filtering technique* to differentiate the large-scale and small-scale modes. This is the same method used in the large-eddy simulation modeling scheme and in our earlier discussion of the 3D case [28]. Here we apply the filter to the scalar equation (1). That is, we consider the ''large-scale scalar field'' defined as the convolution field $\overline{\theta_l}=G_l*\theta$, with some suitable filter function G_l. The resulting equation is

$$\partial_t\overline{\theta_l}(\mathbf{r},t)+\nabla\cdot[\overline{\mathbf{v}_l}(\mathbf{r},t)\,\overline{\theta_l}(\mathbf{r},t)+\mathbf{j}_l(\mathbf{r},t)]=0. \tag{29}$$

The large-scale velocity field is likewise defined by $\overline{\mathbf{v}_l}=G_l*\mathbf{v}$. Note that $\mathbf{j}_l\equiv\overline{(\mathbf{v}\theta)_l}-\overline{\mathbf{v}_l}\,\overline{\theta_l}$ is a space flux of the scalar induced by the turbulent convection (eddy diffusion). A main ingredient of our proofs is the following exact formula for this turbulent flux:

$$\mathbf{j}_l(\mathbf{r},t)=[\Delta\theta(\mathbf{r},t)\Delta\mathbf{v}(\mathbf{r},t)]_l-[\Delta\theta(\mathbf{r},t)]_l[\Delta\mathbf{v}(\mathbf{r},t)]_l. \tag{30}$$

Here $[f]_l = \int d^2\mathbf{s} G_l(\mathbf{s}) f(\mathbf{s})$ is the average over the separation vector $\mathbf{s}$ in the difference operator $\Delta_{\mathbf{s}}$ with respect to the filter function $G_l(\mathbf{s})$. The identity Eq. (30) appeared in [33] and its physical meaning was discussed in [21,28] (see also [34]). Heuristically, $\mathbf{j}_l(\mathbf{r}) \sim \Delta_l \mathbf{v}(\mathbf{r}) \Delta_l \theta(\mathbf{r})$.

Recall that the *scalar-intensity* integral $K(t) = \frac{1}{2}\int_\Lambda \theta^2(t)$ is formally conserved by the full dynamics. From Eq. (29) for the large-scale scalar field it is straightforward to derive by the standard methods of nonequilibrium thermodynamics a local balance equation for its large-scale intensity $K_l \equiv \frac{1}{2}\overline{\theta_l^2}$. It has the form

$$\overline{D_t} K_l(\mathbf{r},t) + \nabla \cdot \mathbf{D}_l(\mathbf{r},t) = -X_l(\mathbf{r},t). \tag{31}$$

Here $\overline{D_t}$ represents $\partial_t + \overline{\mathbf{v}_l}(\mathbf{r},t) \cdot \nabla_{\mathbf{r}}$, the convective derivative associated with the large-scale velocity,

$$\mathbf{D}_l(\mathbf{r},t) \equiv \overline{\theta_l}(\mathbf{r},t) \mathbf{j}_l(\mathbf{r},t) \tag{32}$$

represents the space transport of the large-scale intensity by convective diffusion, and the *scalar flux*

$$X_l(\mathbf{r},t) \equiv -\nabla \overline{\theta_l}(\mathbf{r},t) \cdot \mathbf{j}_l(\mathbf{r},t) \tag{33}$$

represents the scalar transfer to the small-scale modes. In a homogeneous, stationary ensemble the left-hand side of Eq. (31) has a vanishing average. In a steady state with constant mean flux χ of scalar substance to high wave numbers, the average $\langle X_l \rangle = \chi$, a constant, for l lying in the convective interval. Together with Eq. (30), the formula (33) for scalar flux shows that $X_l(\mathbf{r}) \sim \Delta_l \mathbf{v}(\mathbf{r}) [\Delta_l \theta(\mathbf{r})]^2 / l$, which is the RSR, Eq. (10). It is the exact equations (30) and (33) that are the precise form of our RSR, applicable even without assumptions of local isotropy or other statistical properties. They are essentially kinematic in nature, based only upon the conservation properties of the underlying dynamics.

From these exact relations, there follow refinements of the previous exponent inequalities. In fact, it follows from the (generalized) Hölder inequality that for

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{r}, \tag{34}$$

the ordering holds that

$$\begin{aligned} l^{\tau_r(X)/r} &\sim \langle |X_l|^r \rangle^{1/r} \\ &\leqslant \frac{\langle |\Delta_l \theta|^p \rangle^{2/p} \langle |\Delta_l \mathbf{v}|^q \rangle^{1/q}}{l} \\ &\sim l^{2\sigma_p(\theta) + \sigma_q(v) - 1} \end{aligned} \tag{35}$$

and thus

$$2\sigma_p(\theta) + \sigma_q(v) \leqslant 1 + \frac{\tau_r(X)}{r}. \tag{36}$$

This is our main result on the scaling exponents. The derivation requires only the exact kinematical relations Eqs. (30) and (33) rather than the heuristic form of the RSR Eq. (10). For the details of this, see the Appendix of [28]. Note that $r=1$ recovers the previous inequalities Eq. (18) if $\tau_1(X) = 0$. However, as noted above, intermittency of scalar flux will imply that $\tau_r(X) < 0$ for $r > 1$ and then the inequalities are sharpened. This is particularly easy to see for the case of a "synthetic" turbulent convection by a Gaussian random velocity field. For the Gaussian field (or any "monofractal" field) $\sigma_q(v) = h$ for all q. Thus, taking $q \to \infty$ in the above inequality, one easily obtains $r = p/2$ and

$$\zeta_p(\theta) \leqslant \frac{p}{2}(1-h) + \tau_{p/2}(X). \tag{37}$$

Thus $\zeta_p(\theta)$ will be *strictly less* than the "classical exponent" $\zeta_p^{\text{class}} = (p/2)(1-h)$ and the "anomaly" is exactly an intermittency exponent of the convective-range flux. This result implies that convection by a regular-scaling random velocity will nonetheless lead to anomalous scaling for the scalar it passively convects, if the associated flux variable develops strong fluctuations.

It is worthwhile to make a comparison of these results with the other recently proposed RSH for passive scalars [30,31,19]. While our formulation is motivated by the 1974 "revisionist" RSH of Kraichnan, involving flux, the RSH explored by the above authors is an adaptation of that originally proposed by Kolmogorov [32]. That is, it is hypothesized that the random variables

$$V_\theta(\mathbf{r},\mathbf{l}) \equiv \Delta_{\mathbf{l}} \theta(\mathbf{r}) \frac{[l \varepsilon_l(\mathbf{r})]^{1/6}}{[l \chi_l(\mathbf{r})]^{1/2}}, \tag{38}$$

defined in terms of volume-averaged dissipations $\varepsilon_l(\mathbf{r})$ and $\chi_l(\mathbf{r})$ of velocity and scalar intensities, respectively, have conditional distributions given values ε_l and χ_l, which are independent of the local Reynolds number Re_l and local Péclet number Pe_l when those are both large. That is, the variable V_θ is supposed to have a universal distribution in the inertial-convective range of l. If this relation is combined with the original Kolmogorov RSH, then it is easy to infer likewise the existence of a random variable W_θ, universal in the same sense, such that

$$\Delta_l v [\Delta_l \theta]^2 \sim W_\theta \chi_l l. \tag{39}$$

See [19,31]. Given the Kolmogorov RSH for velocity differences, this last relation is, in fact, equivalent to the RSH for passive scalars proposed in [30,31,19]. It provides a natural bridge between that RSH for passive scalars and the one established here. Since we have shown that $X_l \sim \Delta_l v [\Delta_l \theta]^2 / l$, the above relation may be more or less paraphrased as

$$X_l \sim W_\theta \chi_l. \tag{40}$$

In other words, the ratio $X_l / \chi_l \equiv W_\theta$ is a random variable whose distributions conditioned on fixed ε_l, χ_l are universal in the inertial-convective interval of l. Again, this is essentially just a reformulation of the RSH of [30,31]. If it holds, then a simple consequence is that

$$\langle |X_l|^p \rangle = \langle |W_\theta|^p | \varepsilon_l, \chi_l, l \rangle \langle \chi_l^p \rangle \tag{41}$$

and the coefficient $\langle |W_\theta|^p | \varepsilon_l, \chi_l, l \rangle$ is, in the inertial-convective range, just a constant factor $\langle |W_\theta|^p \rangle$. Therefore, in particular, $\tau_p(X)=\tau_p(\chi)$ for all p and the intermittency exponents of the convective-range flux X_l and the scalar dissipation, volume-averaged over the same length scales χ_l, are the same. In that case, all of the inequalities previously rigorously derived in terms of $\tau_p(X)$ hold also for $\tau_p(\chi)$. In terms of providing a theoretical foundation to the RSH for passive scalars, it may be easier to proceed by starting with Eq. (40).

Another interesting comparison involves the *additive fusion rule* (AFR), which was proposed originally for the turbulent velocity gradients [36,37]. Recently, the straightforward extension of these rules to the scalar gradients has received some analytical support in Kraichnan's rapid-change model [3,5–7]. As we now explain, it happens that in this model the AFR and the RSH lead to identical relations between scaling exponents. The AFR states, in a schematic form, that

$$[(\nabla\theta)^{p_1}]\cdots[(\nabla\theta)^{p_n}]\sim[(\nabla\theta)^{p_1+\cdots+p_n}]. \quad (42)$$

We ignore, for the sake of this argument, the vector character of the scalar gradient, which, properly, should be taken into account (cf [36]). The quantities $[(\nabla\theta)^p]$ are so-called renormalized composite variables. This means simply that one defines them as the limit of pth powers of scalar gradients at the same space point, in the model with $\eta_D>0$, but multiplicatively renormalized by an appropriate power of η_D. After nondimensionalizing the variable (according to its "canonical" or engineering dimension) there may still be required a factor $Z(\eta_D)\sim(\eta_D/L)^{-x_p}$ to make the correlations finite in the limit $\eta_D\to 0$. The exponent x_p is the so-called *anomalous scaling dimension* of the variable $[(\nabla\theta)^p]$. Observe, in this context, that the scalar dissipation $\chi(\mathbf{r})=\kappa|\nabla\theta(\mathbf{r})|^2$, when divided by its mean $\overline{\chi}$, is nothing more than $[(\nabla\theta)^2]$. The precise meaning of the schematic result Eq. (42) is that, inserted in arbitrary correlators at separated points,

$$[(\nabla\theta)^{p_1}](\lambda\cdot\mathbf{r}_1)\cdots[(\nabla\theta)^{p_n}](\lambda\cdot\mathbf{r}_n) \sim\lambda^{x_{p_1+\cdots+p_n}-x_{p_1}-\cdots-x_{p_n}}[(\nabla\theta)^{p_1+\cdots+p_n}](\mathbf{0}) \quad (43)$$

in the limit as $\lambda\to 0$. It is easy to show, as in [36], that the above AFR leads to a "multiscaling law" for scalar structure functions (now *without* absolute values) of the form

$$\langle[\Delta_l\theta]^p\rangle\sim l^{(1-x_1)p+x_p}. \quad (44)$$

Moreover, if the short-distance expansion is applied to the moments of the volume-averaged dissipation, it is immediately obtained that

$$\langle\chi_l^p\rangle\sim l^{x_{2p}-px_2}. \quad (45)$$

In Kraichnan's model, $x_1=1-(\gamma/2)$ and $x_2=0$. Thus Eq. (44) leads to $\zeta_p(\theta)=p(\gamma/2)+x_p$ and Eq. (45) yields $\tau_p(\chi)=x_{2p}$. Hence together they give

$$\zeta_p(\theta)=p\frac{\gamma}{2}+\tau_{p/2}(\chi). \quad (46)$$

This should be compared with the result of RSH for the model Eq. (37) [taken as an equality and with $\tau_p(\chi)$ replacing $\tau_p(X)$]. Clearly, they are the same. It was already shown in [7] that the RSH holds in the white-noise model. Our point here is that the RSH is a consequence just of the AFR. This happy situation does not, however, hold for the original AFR applied to velocity gradients, nor even necessarily for scalar gradients in true turbulence. In the case of velocity gradients, the application of the AFR analogous to the above leads to the relation [36]

$$\zeta_p(v)=p\frac{\zeta_2(v)}{2}+\tau_{p/2}(\varepsilon), \quad (47)$$

whereas the RSH leads instead to

$$\zeta_p(v)=\frac{p}{3}+\tau_{p/3}(\varepsilon). \quad (48)$$

These are not equivalent and the RSH result seems to be in better agreement with the experimental data [38]. It is possible to formulate an AFR consistent with the RSH by supposing that the additive algebra Eq. (42) holds with ∇v replaced by the scalar composite $\varepsilon=\nu|\nabla v|^2$ [39]. This is also more plausible in terms of rotational symmetry. As noted in [36], the naive prediction Eq. (47) was based upon a disregard of the tensorial character of products of velocity gradients.

One use of our exact inequalities is as a check on experimental data for scaling exponents, since these employ a variety of assumptions and approximations (Taylor hypothesis, one-dimensional surrogates, etc.). For that purpose, we may cite the experimental results of [14]

$$\zeta_2(\theta)\approx 0.65,\quad \zeta_3(\theta)\approx 0.82,\quad \zeta_4(\theta)\approx 0.95 \quad (49)$$

for the scalar exponents and [35]

$$\sigma_2(v)\approx 0.355,\quad \sigma_3(v)=0.333,\quad \sigma_\infty(v)\leqslant\sigma_{17}(v)\approx 0.211 \quad (50)$$

for the velocity exponents. We shall here assume $\sigma_{17}(v)\approx\sigma_\infty(v)$ since the graph of $\zeta_p(v)$ is close to linear for p of order 20. In that case, if we make comparison, for simplicity, with the Yaglom inequalities, we find that

$$0.65\approx 2\sigma_2(\theta)\leqslant 1-\sigma_\infty(v)\approx 0.78, \quad (51)$$

$$0.55\approx 2\sigma_3(\theta)\leqslant 1-\sigma_3(v)\approx 0.67, \quad (52)$$

$$0.48\approx 2\sigma_4(\theta)\leqslant 1-\sigma_2(v)\approx 0.65. \quad (53)$$

All of these inequalities are well satisfied by the data. The fact that there is a considerable margin between the upper and lower limits is also consistent with an intermittency correction from the scalar flux. Determination of the latter from DNS or, experimentally, from the surrogate $\Delta_l v[\Delta_l\theta]^2/l$, would be of interest.

ACKNOWLEDGMENTS

I would like to thank all the participants of the CNLS Workshop on fully developed turbulence, where most of these results were originally presented, for their comments and questions, but especially R. H. Kraichnan and K. R. Sreenivasan. I also wish to thank G. Falkovich for very helpful discussions regarding his work.

[1] R. H. Kraichnan, Phys. Rev. Lett. **72**, 1016 (1994).
[2] R. H. Kraichnan, V. Yakhot, and S. Chen, **75**, 240 (1995).
[3] A. L. Fairhall, O. Gat, V. L'vov, and I. Procaccia, Phys. Rev. E **53**, 3518 (1996).
[4] K. Gawedzki and A. Kupiainen, Phys. Rev. Lett. **75**, 3834 (1995).
[5] D. Bernard, K. Gawedzki, and A. Kupiainen (unpublished).
[6] M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. E **52**, 4924 (1995).
[7] M. Chertkov and G. Falkovich (unpublished).
[8] M. Chertkov, G. Falkovich, and V. Lebedev, Phys. Rev. Lett. **76**, 3707 (1996).
[9] B. I. Shraiman and E. D. Siggia, Phys. Rev. E **49**, 2912 (1993).
[10] B. I. Shraiman and E. D. Siggia, C. R. Acad. Sci. Paris **321**, 279 (1995).
[11] A. M. Obukhov, Izv. Akad. Nauk. SSSR, Geogr. Geofiz. **13**, 58 (1949).
[12] S. Corrsin, J. Appl. Phys. **22**, 469 (1951).
[13] G. K. Batchelor, J. Fluid. Mech. **5**, 113 (1959).
[14] R. A. Antonia *et al.*, Phys. Rev. A **30**, 2704 (1984).
[15] R. H. Kraichnan, Phys. Fluids **11**, 945 (1968).
[16] G. I. Taylor, Proc. London Math. Soc. **20**, 196 (1921).
[17] A. N. Kolmogorov, Dokl. Akad. Nauk. SSSR **32**, 19 (1941).
[18] A. M. Yaglom, Dokl. Akad. Nauk. SSSR **69**, 743 (1949).
[19] G. Stolovitzky, P. Kailasnath, and K. R. Sreenivasan, J. Fluid Mech. **297**, 275 (1995).
[20] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[21] G. L. Eyink, Physica D **91**, 97 (1996).
[22] P. Constantin and I. Procaccia, Nonlinearity **7**, 1045 (1994).
[23] P. Constantin and I. Procaccia, Phys. Rev. E **47**, 3307 (1993).
[24] U. Frisch and M. Vergassola, Europhys. Lett. **14**, 439 (1991).
[25] L. F. Richardson, Proc. R. Soc. London Ser. A **110**, 709 (1926).
[26] R. H. Kraichnan, J. Fluid Mech. **64**, 737 (1974).
[27] J.-D. Fournier, U. Frisch, and H. A. Rose, J. Phys. A **11**, 187 (1978).
[28] G. L. Eyink, J. Stat. Phys. **78**, 335 (1995).
[29] R. H. Kraichnan, J. Fluid Mech. **62**, 305 (1974).
[30] I. Hosokawa, J. Phys. Soc. Jpn. **62**, 10 (1993).
[31] Y. Zhu, R. A. Antonia, and I. Hosokawa, Phys. Fluids **7**, 1637 (1995).
[32] A. N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962).
[33] P. Constantin, W. E, and E. Titi, Commun. Math. Phys. **165**, 207 (1994).
[34] B. Vreman, B. Geurts, and H. Kuerten, J. Fluid. Mech. **278**, 351 (1994).
[35] J. Herweijer and W. van der Water, Phys. Rev. Lett. **74**, 4651 (1995).
[36] G. L. Eyink, Phys. Lett. A **172**, 355 (1993).
[37] V. V. Lebedev and V. S. L'vov, Pis'ma Zh. Éksp. Teor. Fiz. **59**, 546 (1994) [JETP Lett. **59**, 577 (1994)].
[38] K. Sreenivasan (private communication).
[39] G. L. Eyink (unpublished).